Selected fracture criteria of elastic-plastic materials

O kryteriach pękania materiałów sprężysto-plastycznych

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The paper presents a brief overview of selected fracture criteria of elastic-plastic materials. These criteria, ones of many, can be used to assess the real fracture toughness of different construction materials, provided the user has the required parameters defined by the fracture mechanics.
KEYWORDS: fracture criteria, geometric constraints, J-integral, fracture toughness

The use of elastic-plastic fracture mechanics in engineering analysis is not limited to assessing the distribution of stress and strain fields in front of the crack tip. At present, fracture mechanics also includes: assessment of the level of effort of a structure containing defects (assessment of limit loads, limit load capacity), analysis using appropriate destruction diagrams or assessment of the actual fracture toughness of structural elements with defects [1-3]. While the assessment of the load carrying capacity (relative limit loads) and analysis according to the appropriate destruction diagrams (in the basic approach) [1] do not require knowledge of too many parameters of fracture mechanics characterizing the material of the structural element, it applies to selected fracture criteria in addition to knowledge of basic material constants (yield stress σ_0, Young's modulus E or exponent of the strengthening of n in Ramberg-Osgood relationship), it is necessary to know the parameters of the HRR field [4, 5] and selected measures of geometrical constraints, defined in the works [6, 7].

Among measures of geometrical constraints (i.e. constraints dependent on the geometry and material constants - which construction material puts plastic deformations developing under the influence of external load [6, 7]), one can distinguish: Q stress [8], stress triaxiality parameter T_z [9], mean stresses σ_m normalized by the yield stress or through effective stress (σ_m/σ_0 or σ_m/σ_{eff} respectively) [10]. Parameters of geometrical constraints include also parameters Q^* and Q^*_m [11, 12] and the T_m parameter discussed in the above-mentioned works, i.e. the coefficient of triaxiality of stress T_z, averaged over the thickness of the specimen (structural element) [14]. The values of Q^*_m and T_m, determined as averaged values over the specimen thickness, have already been used to describe the stress fields in front of the crack tip in elastic-plastic materials for 3D issues [13], as well as in the fracture criterion proposed in 2012 [15]. This criterion in the most general form can be saved as:

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$$J_C = J_{IC} \cdot \frac{I_n(n, T_m)}{I_n(n, T_{m\_ref})} \cdot$$

$$\cdot \left[\frac{\widetilde{\sigma}_{ij}(\theta, n, T_{m\_ref})}{\widetilde{\sigma}_{ij}(\theta, n, T_m)} \cdot \left(\frac{\sigma_c - \sigma_0 \cdot Q^*_m(n, T_m)\delta_{ij}}{\sigma_c - \sigma_0 \cdot Q^*_{m\_ref}(n, T_{m\_ref})\delta_{ij}} \right) \right]^{1+n} \quad (1)$$

where:
J_C - actual fracture toughness (expressed using the J-integral [16]);
J_{IC} - critical value of the J-integral, determined experimentally according to the standard [16] for a specimen dominated by a plane strain (in many Eqs. Denoted as "p.s.o.");
$I_n(n, T_m)$, $I_n(n, T_{m\_ref})$, $\widetilde{\sigma}_{ij}(\theta, n, T_m)$, $\widetilde{\sigma}_{ij}(\theta, n, T_{m\_ref})$ - functions that can be determined on the basis of literature data or using the computer program presented in [17] for the T_m parameter value (characterizing the triaxiality level of stress for the analyzed element) or T_{m_ref} parameter (characterizing the triaxiality level of stresses for the specimen, for which J_{IC} has been determined);
θ - coordinate in the polar coordinate system (r, θ, z), hooked at the tip of the crack in the axis of the specimen (structural element);
Q^*_{mi} and Q_{m_ref} - averaged over the thickness of the Q^* [11-13] parameter value, for the considered structural component for which J_C fracture toughness is sought, and the specimen for which the J_{IC} value was determined experimentally (Q^*_m and Q_{m_ref} parameters depend on the strain hardening exponent n and the value of the parameter T_m);
σ_c - critical stresses, which can be determined for a given material on the basis of laboratory experiments and numerical calculations, or can be found in the literature [18];
δ_{ij} - Kronecker delta.

Fracture criterium of elastic-plastic materials (1) have been discussed in detail in [15], where it was verified using commonly accepted experimental results presented in [19]. It is worth noting that the criterion presented in [15] is true for three-dimensional structural elements, while the criteria presented in [8, 18] relate to a plane strain state.

Based on the method of determining the fracture criterion described in [15], analogous approaches can be proposed using other parameters of geometrical constraints, such as:
- quotient of normal average stresses σ_m and yield point (yield stress) σ_0:

$$\frac{\sigma_m}{\sigma_0} = \frac{1}{\sigma_0} \cdot \frac{(\sigma_{11} + \sigma_{22} + \sigma_{33})}{3} \quad (2)$$

where: σ_{11}, σ_{22}, σ_{33} - normal components of stress tensor;

- quotient of normal mean stresses σ_m and effective stresses σ_{eff} according to the HMH hypothesis - σ_m/σ_{eff};
- Q^{pso} parameter being the normalized by the yield point difference between the actual stress distribution σ_{22_MES} and the distribution estimated for the plane strain σ_{22_pso}:

$$Q^{\mathrm{pso}} = \frac{\sigma_{22\_\mathrm{MES}} - \sigma_{22\_\mathrm{pso}}}{\sigma_0} \tag{3}$$

where: $\tag{4}$

$$\sigma_{ij\_\mathrm{pso}} = \sigma_0 \cdot \left(\frac{J}{\alpha \cdot \sigma_0 \cdot \epsilon_0 \cdot I_n(n, \mathrm{pso}) \cdot r} \right)^{\frac{1}{1+n}} \cdot \widetilde{\sigma_{ij}}(\theta, n, \mathrm{pso})$$

whereby: $I_n(n, \mathrm{pso})$, $\widetilde{\sigma_{ij}}(\theta, n, \mathrm{pso})$ - functions determined according to the algorithm presented in [13, 17] assuming the domination of the plane strain state.

These parameters should always be determined for three-dimensional structural objects (specimens) in order to include both in-plane and out-of-plane constraints in the proposed fracture criteria [6, 7]. Next, proposals for new fracture criteria for elastic-plastic materials together with the assessment of the impact of various parameters of fracture mechanics are discussed.

A two-parameter fracture criterion $J = J(J_{IC}, Q_m^{pso})$

As already mentioned, one of the measures of geometric constraints is the Q^{pso} parameter. Some researchers [20, 21] proposed the expression of this parameter using the Q stress formula, defined by O'Dowd [8], as the amount correcting the stress distribution obtained on the basis of the HRR solution [5, 6]. This parameter should be counted at each point in front of the crack tip, along its entire front. An alternative formula for the stress distribution before the crack tip for elastic-plastic materials in the case of 3D issues has the form:

$$\sigma_{ij} = \sigma_0 \cdot \left(\frac{J}{\alpha \cdot \sigma_0 \cdot \epsilon_0 \cdot I_n(n, \mathrm{pso}) \cdot r} \right)^{\frac{1}{1+n}}$$
$$\cdot \widetilde{\sigma_{ij}}(\theta, n, \mathrm{pso}) + Q^{\mathrm{pso}} \cdot \sigma_0 \cdot \delta_{ij} \tag{5}$$

It is proposed that the Q^{pso} parameter should be determined for the direction $\theta = 0$ and the distance $r = 2J/\sigma_0$. Formula (5) is true for each point in front of the front of the crack. This description in the case of a structural element of a specified thickness and resistance to cracking takes the form:

$$\sigma_{ij} = \sigma_0 \cdot \left(\frac{J}{\alpha \cdot \sigma_0 \cdot \epsilon_0 \cdot I_n(n, \mathrm{pso}) \cdot r} \right)^{\frac{1}{1+n}} \cdot \widetilde{\sigma_{ij}}(\theta, n, \mathrm{pso})$$
$$+ Q^{\mathrm{pso}}_{\mathrm{av}} \cdot \sigma_0 \cdot \delta_{ij} \tag{6}$$

where: $Q^{\mathrm{pso}}_{\mathrm{av}}$ - the value of the Q^{pso} parameter, averaged over the thickness.

When formulating the criteria for cracking elastic-plastic materials that are characterized by a brittle fracture mechanism, it is assumed that the cracking process takes place when the crack opening stress reaches the critical value σ_c at a specified critical distance from the crack tip, denoted by r_c [15, 19, 22]. This type of analysis is presented in detail in [15, 19, 22]. Using the analysis method presented in [15], an alternative to [15] crack criterion can be given by analogy, based on the $Q^{\mathrm{pso}}_{\mathrm{av}}$ parameter:

$$J_\mathrm{C} = J_\mathrm{IC} \cdot \left[\frac{\sigma_\mathrm{c} - Q^{\mathrm{pso}}_{\mathrm{av}} \cdot \sigma_0 \cdot \delta_{ij}}{\sigma_\mathrm{c} - Q^{\mathrm{pso}_{\mathrm{ref}}}_{\mathrm{av}} \cdot \sigma_0 \cdot \delta_{ij}} \right]^{n+1} \tag{7}$$

where: J_C - actual sought fracture toughness; $Q^{\mathrm{pso}}_{\mathrm{av}}$ - parameter determined for the specimen for which J_C is sought (you can use the catalog of ready solutions given in [21]); $Q^{\mathrm{pso}_{\mathrm{ref}}}_{\mathrm{av}}$ - the value of the parameter $Q^{\mathrm{pso}_{\mathrm{ref}}}_{\mathrm{av}}$ determined numerically for the specimen for which the critical value of fracture toughness J_{IC} was determined for the laboratory; σ_c - critical stress.
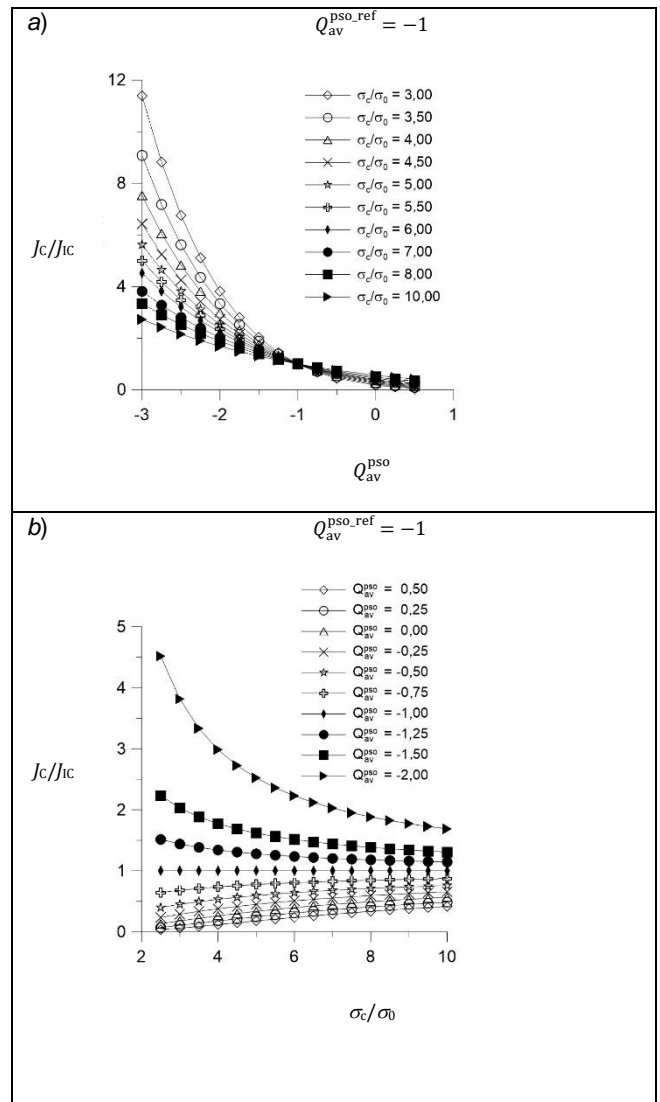

Fig. 1. Effect of: a) parameter $Q^{\mathrm{pso}}_{\mathrm{av}}$ (for different values of critical stresses σ_c/σ_0) to the value of the real fracture toughness; b) critical stresses σ_c/σ_0 (for different values of parameter $Q^{\mathrm{pso}}_{\mathrm{av}}$) to the value of the real fracture toughness

Fig. 1 depicts hypothetical changes in the real fracture toughness J_C normalized by fracture toughness J_{IC} determined with the assumption of domination of planes train state. The higher the value of critical stresses σ_c, the smaller the value of the real fracture toughness J_c, determined for the set values of the parameters of ties

Q_{av}^{pso} and $Q_{av}^{pso_{ref}}$. However, it should be noted that at a given level of Q_{av}^{pso} and $Q_{av}^{pso_{ref}}$ parameters, this trend is slightly reversed. The higher the value of parameter Q_{av}^{pso} the lower the real fracture toughness J_C. It can be concluded that the higher the level of constraints (the more negative the Q_{av}^{pso} parameter), the higher the real fracture toughness J_C.

A two-parameter fracture criterion $J = J(J_{IC}, \sigma_m/\sigma_0)$

In 1995, O'Dowd [8], summing up the previous considerations regarding Q stress, suggested that they should be determined not only taking into account the stress opening the crack surfaces, but as a parameter dependent on the mean stresses σ_m:

$$Q_m = \frac{\sigma_{m\_MES} - \sigma_{m\_pso}}{\sigma_0} \text{ dla } \theta = 0 \text{ i } r = 2J/\sigma_0 \quad (8)$$

where subscript „MES" denotes that stress components were calculated using Finite Element Method, but subscript „pso" denotes, the stress components where calculated using HRR solution for plane strain state.

If, therefore, the Q stress in the fracture criterion proposed by O'Dowd [8] is replaced by the parameter Q_m, it can be written in a new form:

$$J_C = J_{IC} \cdot \left[1 - \frac{Q_m \cdot \sigma_0}{\sigma_c}\right]^{n+1} \quad (9)$$

Development of equation (9) leads to the following form of fracture criterion:

$$J_C = J_{IC} \cdot \left[1 - \frac{\sigma_0}{\sigma_c} \cdot \left(\frac{\sigma_m}{\sigma_0} - \frac{\sigma_{m\_pso}}{\sigma_0}\right)\right]^{n+1} \quad (10)$$

Formula (10) can be used when the critical value of J_{IC} fracture toughness is determined in the laboratory for a specimen dominated by plane strain state, for which the crack length according to the recommendations of the standard is $a/W = 0.5$, and the value of parameter $Q_m = 0$.

However, under real laboratory conditions, although the crack length is $a/W = 0.5$, the materials of such specimens are extremely rarely characterized by a bond level of zero. This fact should be taken into account in the fracture criterion, introducing an appropriate reference condition.

Based on these considerations, a new form of the fracture criterion (10) can be written by analogy, in which the reference state will be characterized by the value of parameter $Q_m \neq 0$.

$$J_C = J_{IC} \cdot \left[\frac{\sigma_c - \sigma_0 \cdot \left(\frac{\sigma_m}{\sigma_0} - \frac{\sigma_{m\_pso}}{\sigma_0}\right)}{\sigma_c - \sigma_0 \cdot \left(\frac{\sigma_{m\_ref}}{\sigma_0} - \frac{\sigma_{m\_pso}}{\sigma_0}\right)}\right]^{n+1} \quad (11)$$

The results presented in [21] in the form of changes in the parameter value σ_m/σ_0 may facilitate the use of fracture criteria (10) and (11). Fig. 2 presents the influence of various parameters on the real fracture toughness J_C, which were used in the criterion proposal (11). The greater the level of average stress σ_m/σ_0, the smaller the value of the real fracture toughness J_C, which also decreases with the increase of the level of critical stress σ_c.
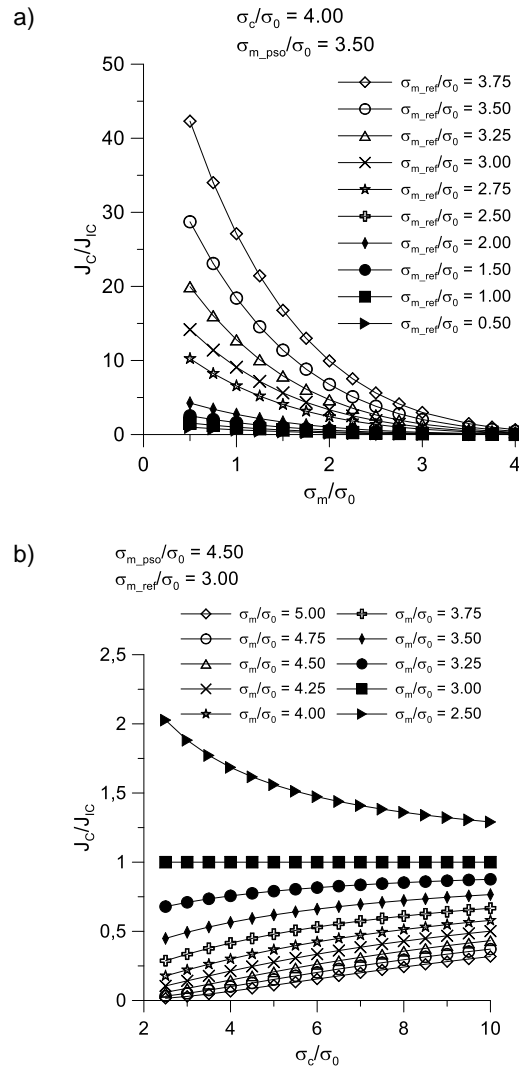

Fig. 2. Influence of the selected parameters in the formula (11) on the real fracture toughness: a) the effect of the quotient σ_m/σ_0 for the determined values of σ_{m_pso}/σ_0 and σ_c/σ_0; b) the impact of the quotient σ_c/σ_0 for the set values σ_{m_pso}/σ_0 and σ_{m_ref}/σ_0

A two-parameter fracture criterion $J = J(J_{IC}, \sigma_m/\sigma_{eff})$

In 1995 Guo Wanlin [14] presented the following fracture criterion:

$$J_C \cdot F\left(\frac{\sigma_m}{\sigma_{eff}}\right) = \text{const} \quad (12)$$

but, in the form of $F\left(\frac{\sigma_m}{\sigma_{eff}}\right)$ function, the influence of out-of-planes constraints (the constraints in the thickness direction) is taken into account by means of the triaxiality coefficient of stresses:

$$F\left(\frac{\sigma_m}{\sigma_{eff}}\right) = \frac{2}{3}(1 + \nu) + 3(1 - 2\nu)\left(\frac{\sigma_m}{\sigma_{eff}}\right)^2 \quad (13)$$

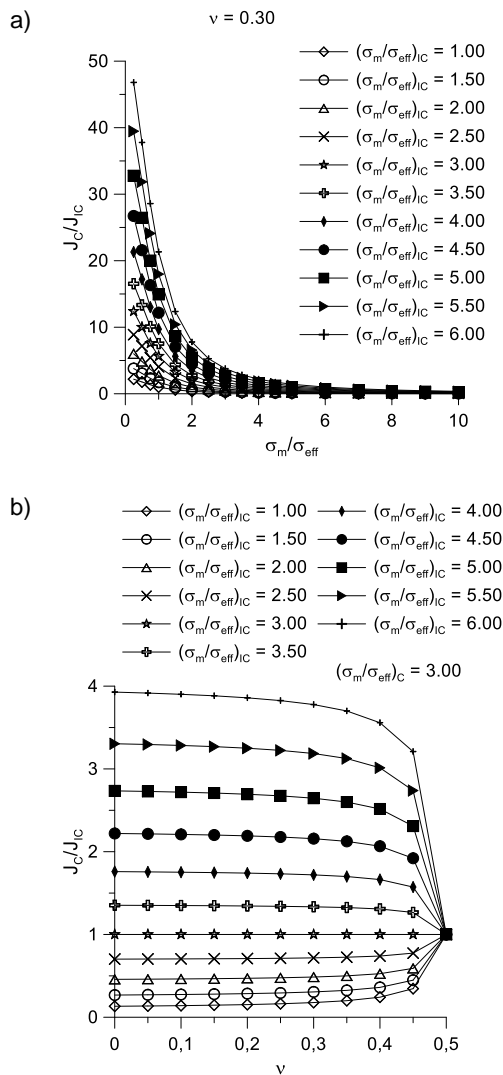where: ν - Poisson's ratio.

a)

b)

Fig. 3. Influence of the selected parameters in the formula (14) on the real fracture toughness: a) the effect of the quotient σ_m/σ_{eff} for the determined values $(\sigma_m/\sigma_{eff})_{IC}$; b) the effect of Poisson's coefficient on set values $(\sigma_m/\sigma_{eff})_{IC}$

Based on the formula (12), one can postulate that the real fracture toughness J_C be calculated as:

$$J_C = J_{IC} \cdot \left[\frac{\frac{2}{3}(1+\nu) + 3(1-2\nu)\left(\frac{\sigma_m}{\sigma_{eff}}\right)^2_{IC}}{\frac{2}{3}(1+\nu) + 3(1-2\nu)\left(\frac{\sigma_m}{\sigma_{eff}}\right)^2_{C}} \right] \qquad (14)$$

where the subscripts mean: *IC* - that the value of the J-integral or the quotient σ_m/σ_{eff} was determined for the spcimen for which the critical fracture toughness J_{IC} was determined according to the ASTM standard; *C* - that the value of the J-integral or the quotient σ_m/σ_{eff} refers to the specimen for which the real fracture toughness J_C is sought.

Numerical calculation results given in [21] in the form of changes in the parameter value σ_m/σ_{eff} may facilitate the use of the fracture criterion (14). Fig. 3 shows the influence of selected parameters on the value of the real fracture toughness. The higher the value of the quotient σ_m/σ_{eff}, the smaller the value of the real fracture toughness J_C, which decreases with the increase of the Poisson coefficient, if the quotient $\sigma_m/\sigma_{eff} < 3$.

A two-parameter fracture criterion $J = J(J_{IC}, d_n)$

The fracture criterion proposals presented above are suitable for materials characterized by a brittle fracture mechanism, when the cracking process begins after the stress has reached the critical value σ_c at a critical distance from the cracks apex r_c. In the case of considerations regarding the process of cracking of ductile materials, the crack tip opening displacement is a convenient parameter [7]. In 1995, O'Dowd [8] mentioned the fracture criterion based on this particular fracture parameter. He assumed that the cracking process would occur if the crack tip opening displacement reached the critical value δ_c. In 1981, Shih connected the crack tip opening displacement δ_T with the J-integral value:

$$\delta_T = d_n \cdot J \qquad (15)$$

where: d_n - coefficient determined for a given yield point quotient σ_0/E, strain hardening exponent n, and also for plane stress ar plane strain.

The d_n coefficient is determined when determining the parameters of the function with the tilde "~", while solving the differential equation of the fourth degree as part of the analysis of the HRR problem. This function can be estimated using the computer program indicated in the works [13, 17]. Guo Wanlin [14], the value of the d_n function additionally depended on the triaxiality coefficient of stress denoted as T_z - the aforementioned program determines the value of the d_n coefficient also for such an approach. Fig. 4 presents changes in the value of the d_n coefficient in the function of the yield point quotient and Young's modulus for the four values of the Ramberg-Osgood's exponent in the case of plane strain state domination [24].
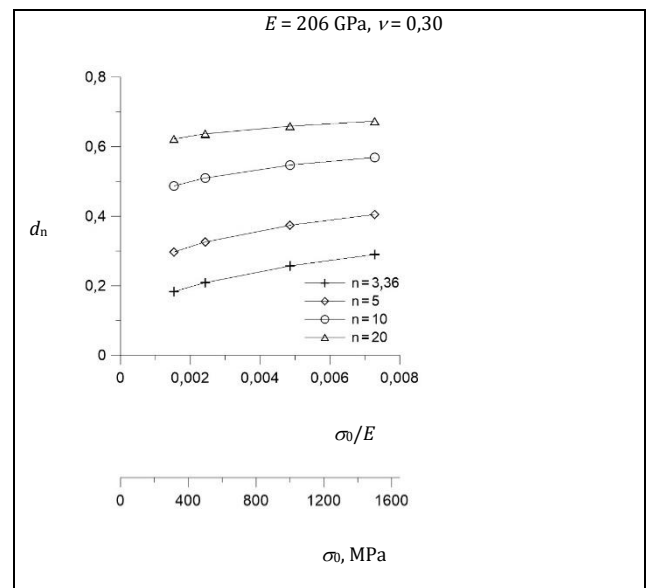

Fig. 4. Values of d_n coefficient (15) for various strain hardening exponents and yield stress [24]

The numerical tests carried out indicate, however, that the value of the d_n coefficient depends not only on the material, but also on the geometry of the specimen (fig. 5a). In work [24] the dependence of the d_n coefficient on the Q stress in the case of the SEN(B) beam dominated by plane strain. is proved, as well as the dependence of this coefficient on the level of external load [24]. As it can be seen, after exceeding a certain level of external load, the value of d_n coefficient reaches the saturation level and does not change with the increasing external load (fig. 5b) [24].

For the case of plane strain, O'Dowd's proposed cracking hypothesis for a normative specimen with a relative crack length of $a/W = 0.5$ can be represented as:

$$\delta_c = d_n^{IC}(n, \sigma_0/E, a/W) \cdot J_{IC} \tag{16}$$

In turn, for any specimen that does not meet the normative requirements, but made of the same material, it can be written that:

$$\delta_c = d_n^{C}(n, \sigma_0/E, a/W) \cdot J_C \tag{17}$$

The combination of both models leads to the formulation of the fracture criterion in the form of:

$$J_C = J_{IC} \cdot \frac{d_n^{IC}(n, \sigma_0/E, a/W)}{d_n^{C}(n, \sigma_0/E, a/W)} \tag{18}$$

where: d_n^{IC} - value of d_n coefficient for the specimen for which critical fracture toughness J_{IC} which was determined in laboratory conditions; d_n^{C} - value of d_n coefficient for the specimen for which the real fracture toughness J_C is sought.

The use of the fracture criterion (18) first of all requires knowledge of fracture toughness determined for planes train state - J_{IC}, but also values of d_n^{IC} and d_n^{C} In [24], values of the $d_n = f(n,\sigma_0/E, a/W)$, coefficient were estimated based on numerical calculations, and the obtained results were approximated to facilitate possible engineering analyzes.

The formula (19) allows to estimate the value of the d_n coefficient provided that the characteristics of the material are known (yield point σ_0, strain hardening exponent n) and the relative crack length a/W [24]:

$$d_n(n, \sigma_0/E, a/W) = a_1 + b_1 \cdot \ln(\sigma_0/E) + c_1 \cdot (n)^{d_1} \tag{19}$$

where: a_1, b_1, c_1 and d_1 - coefficients that are functions of the relative crack length a/W, which can be estimated as [24]:

$$a_1(a/W) = -5{,}6957 \cdot (a/W)^3 + 8{,}551 \cdot (a/W)^2 + \\ - 4{,}241 \cdot (a/W) + 1{,}5734$$

$$b_1(a/W) = 0{,}011 \cdot (a/W)^3 + 0{,}0191 \cdot (a/W)^2 + \\ - 0{,}0508 \cdot (a/W) + 0{,}0652 \tag{20}$$

$$c_1(a/W) = -8{,}2892 \cdot (a/W)^3 + 10{,}394 \cdot (a/W)^2 + \\ - 3{,}2233 \cdot (a/W) - 1{,}242$$

$$\begin{cases} d_1(a/W) = -3{,}3333 \cdot (a/W) + \\ \quad - 0{,}3333 & \text{dla } a/W = \langle 0{,}05; 0{,}20 \rangle \\ d_1(a/W) = -1 & \text{dla } a/W = \langle 0{,}20; 0{,}70 \rangle \end{cases}$$

Presented method of analysis is not, in principle, either revealing or new, but sometimes when assessing fracture toughness, the fracture criterion given in 1995 by O'Dowd is omitted [8], mainly due to the difficulty in determining various parameters. The criterion is very obvious and easy to apply if values of the d_n coefficient, are known [24]. The use of criterion (18) leads to obtaining safe results (which O'Dowd mentioned in 1995). This criterion was developed by the authors of the work [25], in which the method of taking into account geometrical constraints in the case of determining the real fracture toughness was presented.
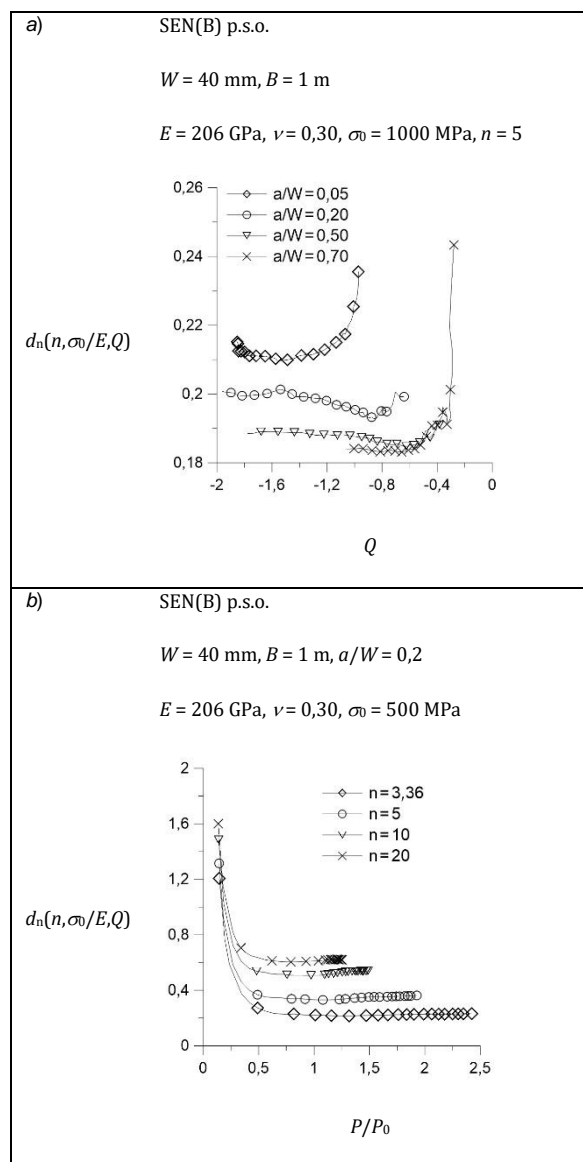
Fig. 5. The effect of: a) the relative crack length a/W on the value of $d_n(n, \sigma_0/E, Q)$ as a function of the level of in-plane constraints expressed in Q [24]; b) the exponent n in the Ramberg-Osgood law on the value of the coefficient $d_n(n, \sigma_0/E, Q)$ as a function of the external load P normalized by the limit load P_0 [24]

Conclusions

The paper presents a short overview of selected fracture criteria for cracking elastic-plastic materials. Some of the presented models - (1), (12) and (18) - were published in [15, 14, 8]. The remaining ones were created as a result of a trivial analysis based on the O'Dowd's idea from 1995 [8]. It may seem that the fracture criteria presented are simple, but - as shown in [15, 19] - uncomplicated solutions can lead to correct, and above all, safe results that are expected in the case of assessing the strength of a structure containing defects.

Based on the presented information, the reader can individually derive the fracture criteria proposed in the work and assess - based on the presented drawings - how the fracture toughness changes with the change of geometrical constraints (expressed by means of various parameters) or reference states.

Although many works have been devoted to the criteria of fracture [6-8,14, 15, 18, 19, 22, 25], the subject of assessing the actual fracture toughness still remains open, especially in the search for new and above all uncomplicated solutions.

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